1. Solving the following differential equations.

(a) (10%) $x^2y'' - 2y' - y = 0, y(x) = ?$

(b) (10%) $y'' + y' - 6y = 6xe^{2x}$, $y(x) = y_h + y_p$; y_h is the general (or complementary) solution, y_p is the particular solution. Find $y(x) = ?$.

2. The definition of the "unit step function" $u(t-a)$ is $u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$.

(c) (10%) If $F(s) = \mathcal{L}\{f(t)\}$ and a>0, please prove that $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$.

(d) (10%) Solve $y' + y = f(t), y(0) = 2$ where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \cos t, & t \ge \pi \end{cases}$.

3. (10%) The power series $y(x) = \sum_{n=0}^{\infty} c_n x^n$ is the solution of $y'' - 10xy = 0$. Find the coefficients c_2, c_3, c_4, c_5, c_6, c_7, and c_8 in values or in terms of c_0 and c_1.

4. (10%) $f(t) = \begin{cases} 1 & -1 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$ and $f(t) = f(t+3)$, derive its Fourier series.

(Hint: $f(t) = f(t+T)$, $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T})$

where $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt,\ a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt,\ b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt$)

5. (10%) $f(t) = \begin{cases} 2 & 0 \le t < 1 \\ 0 & else \end{cases}$ derive its Fourier Transform. (Hint: $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$)

6. (10%) Given $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, solve A^{100} .

7. $y'' + 3y' + 2y = 1$ with $y(0) = 0, y'(0) = 0$

(a) (5%) derive its state space representation with $x(t) = [y(t)\ y'(t)]^T$ in the form $x' = Ax + Bu(t)$.

(b) (5%) calculate e^{At}. (Hint: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$)

(c) (10%) solve $x(t)$. (Hint: $x(t) = e^{At}x(0) + \int_0^t e^{A\tau}Bu(t-\tau)d\tau$)