行政院國家科學委員會專題研究計畫 成果報告

線性彈性體與 Stokes 方程奇異問題與數值方法的研究
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計 畫 主 持 人：李明恭

計畫參與人員：碩士班研究生-兼任助理人員：蔡其璋
博士班研究生-兼任助理人員：宋睿偉

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中 華 民 國 101 年 10 月 05 日

中文摘要：線性彈性體的邊界角落(corners)及裂縫的奇異問題(crack singularity)的計算不管在實務驗證或是理論上都十分重要，這些角落及裂縫奇異問題及奇異的特徵性質的結果對於能否設計出有效的數值方法具有重要的關鍵，然而使用傳統的有限元(FEM)及有限差分法(FDM)或是只使用基本解(fundamental solutions)法這幾種數值方法處理這類型的問題都有精確度不夠的基本問題存在。我們的研究案基本上是希望能對線性彈性體設計新的邊界角落及裂縫奇異模型及它們的精確數值解。嘗試由滿足線性彈性體的一般特別解(general particular solutions)直接推導出奇異特別解(singular particular solution)，藉此方式建立一個系統性的求解程序，在計算上可以使用包含基本解及特別解，或是基本解與少數奇異特別解所形成的計算方式針對一般平滑邊界或具有奇異角點或奇異邊界的問題作計算。

中文關鍵詞：線性彈性體、基本解、特別解、奇異特別解、Trefftz 法、邊界元法

英文摘要：The singular solutions for linear elastostatics problems are important in both theory and computation. The singular property and the singular solutions near corners are both important to design effective numerical schemes. Traditional finite element and finite difference methods provide poor accuracy of numerical solutions for singularity, unless increasing re-meshing strategy is used, but the cost of computation is high.

Our research is to set up a systematic analysis for linear elastostatics with corner or edge singularity. Our effort is to build a systematical strategy to derive singular particular solutions from general particular solutions depending on different boundary condition, such as mixed boundary conditions on different edges or mixed boundary conditions on the same corner edge. It is a continued work from our previous articles which deal with seeking the general singular solutions of linear elastostatics near the corners under free traction boundary conditions. In addition, we will design more effective models of crack singularity under mixed boundary conditions. The particular solutions and fundamental solutions can be used for plane strain and plane stress problems, to lead to the method of particular solutions (MPS) and the method of fundamental solutions (MFS), respectively. For the crack models, we will develop the combined Trefftz method (TM) which is with a few singular particular solutions (PS) and many fundamental solutions (FS), to well suit for linear elastostatics with corners.

英文關鍵詞： linear elastostatics, fundamental solutions, particular solutions, singular particular solutions, Trefftz methods, collocation Trefftz method, boundary elements method

行政院國家科學委員會補助專題研究計畫 □期中進度報告
■期末報告

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計畫主持人：李明恭教授
共同主持人：
計畫參與人員：宋睿偉、蔡其璋、莊美惠

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□移地研究心得報告
□出席國際學術會議心得報告
□國際合作研究計畫國外研究報告

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中 華 民 國 101 年 10 月 15

前言：

線性彈性體的邊界角落(corners)及裂縫的奇異問題(crack singularity)的計算不管在實務驗證或是理論上都十分重要，這些角落及裂縫奇異問題及奇異的特徵性質的結果對於能否設計出有效的數值方法具有重要的關鍵，然而使用傳統的有限元(FEM)及有限差分法(FDM)或是只使用基本解(fundamental solutions)法這幾種數值方法處理這類型的問題都有精確度不夠的基本問題存在。我們的研究案基本上是希望能對線性彈性體設計新的邊界角落及裂縫奇異模型及它們的精確數值解。嘗試由滿足線性彈性體的一般特別解(general particular solutions)直接推導出奇異特別解(singular particular solution)，藉此方式建立一個系統性的求解程序，在計算上可以使用包含基本解及特別解，或是基本解與少數奇異特別解所形成的計算方式針對一般平滑邊界或具有奇異角點或奇異邊界的問題作計算。在這個計畫之前，我們已經有一些基礎的工作完成，發表在Engineering Analysis with Boundary Elements, Vol. 34, pp. 533~548. **[SCI] (IF:1.531-1.704-5 years),** 該文討論的線彈性體的邊界條件是夾緊(Clamped boundary condition) and free traction (與之前相異的條件)兩種。它們的特解及部分代數顯示奇異特別解都已經推導出，包含兩組內裂縫及邊界裂縫模型已經建立，因為這些模型已經有非常精準的數值解，所以根據我們的方法

(collocation Trefftz methods)所得的結果可以互相比較，在我們使用雙精確所計算出來的leading coefficient (亦即Stress intensity factor K_I) 有14 (or 13) 有效數字。在這個研究案提出之後及進行期間，我們又陸續依照這第一篇文章所訂下的基礎,發展在其他不同邊界條件下的特別解,陸續發表在相同的國際期刊總共達四篇之多。以下介紹其他的論文。第二篇論文我們所探討的邊界條件分別是：無牽引力(free traction conditions), 這篇論文是前面論文的延續,也是 Neumann問題的表述。這組線彈性體的邊界條件是free traction。同樣它們的特解及部分代數顯示奇異特別解都已經推導出,包含兩組內裂縫模型已經建立(包含對稱與反對稱兩種)。除了系統化的求其裂縫奇異特別解的過程，我們亦設計出基本解與奇異特別解合併，組成一個 Combined Trefftz method特別針對Cauchy-Navier equation計算。部分結果顯示如果有基本解加上一些我們所推導出來符合部份traction邊界條件的奇異解，那麼我們可以將crack計算的很精準，比起文獻中只使用基本解法加上一項奇異解要好許多。這篇論文表在Engineering Analysis with Boundary Elements, Vol. 34, pp 632~654, 2010. [SCI]。

再來是第一種混合邊界條件，位移(displacement)及無牽引力條件(free traction)在相同的邊上,發表在 Engineering Analysis with Boundary Elements, Volume 35, Issue 12, December 2011, Pages 1265-1278. [SCI] (IF:1.531-1.704-5 years)；最後一種是第二型混合邊界條件位移

(displacement)及無牽引力條件(free traction)但是發生在不同邊上，發表在 Engineering Analysis with Boundary Elements, Vol. 36 (2012), pp. 1125-1137. (2012.03.05) [SCI] (IF:1.531-1.704-5 years)。我們對於線彈性體的研究已經將可能發生的不同邊界條件下的奇異特別解推導出來，在計算方面我們使用 Trefftz Collocation method，並應用混合法，就是在非角點時使用基本解，在接近角點時則使用些許奇異特別解，這樣的計算結果我們也與另一個使用方法做比較，就是在非角點區域使用一般連續型態的特別解，在接近角點時也使用推導出之奇異特別解，但是計算結果發現使用基本解加上些許奇異特別解的收斂速度最快最好，原因出現在一般型特別解與奇異特別解在某些方面性質過於接近而降低了他們的功效，與使用性型態的基本解加上一些奇異特別解的混合方法，作者強烈推薦工程師使用後者在具有角點的線彈性體使用我們推導的奇異特別解加上基本解是最好的。

研究目的：

一個具有奇異性的線彈性體問題的計算，在我們所著重討論的情形是具有奇異角點的狀況，其他當然有不同的奇異狀況，例如在一個邊上有裂縫則不在我們討論的情形下;如何能有系統的將基本解與奇異特別解合併在計算的演算法中去處理線性彈性體奇異邊界與角落是這

個計畫的研究重點之一。因為基本解在非奇異裂縫處是平滑(smooth)的，但是基本解對於奇異問題的計算一般而言數值結果則比較不準，因為有奇異點(singularity)的影響，所以單獨使用基本解的算法無法滿足數值解準確性的要求；另一方面在計算中除了少數某些裂縫尖端可以直接寫出奇異特別解的顯式表示式以外，其他的角落，例如 L 形的角落，部份解的特徵值參數必須藉由其他數值方法事先求得，所以這些類型的角落特別解都比較麻煩處理，所以使用太多的奇異特別解在計算上也是不便之處。因此在這個研究中針對奇異裂縫處我們著重設計出如何只使用一些基本解及少數項的奇異特別解的計算模式而形成一個稱為混合 Trefftz 方法(Combined Trefftz method)的計算模式，希望能夠處理具邊界角落及裂縫的奇異問題的計算是我們的研究目的，這樣的計算方法也可以推廣給工程師在計算奇異線彈性體時使用。另一個目的當然是規劃許多不同的邊界條件，包括 Dirichlet, Neumann, Robin boundary conditions. 在這裡我們使用的是一種混合邊界條件，一個是 displacement and free traction 在同邊上發生，另一個則是 displacement and free traction 發生在不同邊上的情形，我們將在研究方法中說明。

文獻探討：

線性彈性體的角落(corners)及裂縫奇異問題(crack singularity)的計算不管在實務或是理論驗證上都十分重要，這些角落及裂縫奇異問題及奇異的特徵值的結果對於能否設計出有效的數值方法具有重要的關鍵，然而傳統的有限元(FEM)及有限差分法(FDM)處理這類型的問題都有精確度不夠的問題存在。許多提昇精確度及穩定性的計算方法在許多類型的問題中陸續提出，這些方法之中尤其在 Poisson’s equation 及其他橢圓偏微分方程，[Li, 1998]李提出許多系統化的解決方案。一般處理裂縫奇異問題的計算，對於奇異性質的模型選定或是設計在程序模擬上很重要，在 Poisson’s equation 中 Motz’s 問題是裂縫奇異問題(crack singularity)的一個標準模型，它的精確奇異特別解在 [Li,1998], [Li, 1987], [Lu, Chang, Huang, Li, 2009]中使用 Collocation Trefftz Method(CTM)已經提出。在 biharmoniic equation 的問題，類似的 Motz’s 模型在[Li, Lu, Hu, 2004]也第一次提出，其穩定性分析則在 [Li, Lu, Wei, 2009] 另外提出。另外在給定不同邊界限制的條件下的情形，如接近角落(near corners)在沒有 stress 邊界條件(free stress boundary conditions)下，[Lin, Tong, 1980]，[Qin, 2000]等人都有提出使用不同方法，包括 FEM, Hrbrid Trefftz 等的結果。2009 年我們在 [Li, Chu, Young, Lee, 2010]也針對線性彈性體(linear elastostatics) 當發生

夾緊(clamped)及 free stress boundary condition 下推導奇異特別解(particular solutions)，尤其需要特別指出的是我們推導出來的奇異特別解在 free stress boundary condition 下其顯式結果與 [Lin, Tong, 1980] 的結果相同。在我們文中又針對 Motz’s 模型設計適用在 clamped 及 free stress boundary condition 兩種狀況下的裂縫模型，包括對稱與反對稱的兩種狀況，這些模型也是第一次出現在 linear elastostatics 文獻之裂縫模型。這個研究案也會是研究裂縫奇異問題使用 collocation Trefftz Method(CTM)的計算研究。當然 CTM 也可視為 boundary element method (BEM)的另一種型態的變化。最基礎的狀況是奇異特別解(singular particular solutions) 在整個計算過程取代基本解，並且只要依據邊界條件的設定推導，並儘量滿足邊界條件，最後利用最小邊界能量函數(boundary energy)，並使用數值積分方法計算。所以 CTM 最奇特之處是在於將偏微分方程由問題的定義範圍改變到只處理邊界問題而已，所以整個計算量及資料儲存量可以大大減少。但是優點也是它的缺點，就是奇異特別解必須預先計算知道，在大部分條件下是需要以數值方法求特徵值，但是也可以在某些特殊邊界角落以顯示(explicit)表示。所以我們的研究案基本上是希望能對線性彈性體尤其是 Cauchy-Navier equation 推導奇異特別解作一個系統性的分析及建構方法，並且嘗試由滿足線性彈性體的 Cauchy-Navier equation

的一般特別解解直接推導出滿足新的邊界條件的奇異特別解，並藉此方式建立一個系統性的奇異問題的求解程序。當然我們的結果仍然必須與已知文獻，例如：[Lin, Tong, 1980]，[Qin, 2000]等相同才行。

另外針對具裂縫奇異模型(crack singularity)的計算在混合邊界條件下，嘗試找更有效的模型，當然如[Lu, Hu, Li, 2004]所提模仿 cracked-beam 的模式之 Motz's 模型，並利用 collocation Trefftz method (CTM) 方法解決。還有另一個重要的問題是在具有奇異裂縫處，如何能有系統的將基本解與特別解合併去處理線性彈性體角落或奇異裂縫問題(linear elastostatics corners, crack singualrity)。因為基本解在非奇異裂縫處還是平滑的函數，但是基本解對於奇異問題的計算一般而言數值結果會比較不準。另一方面在計算中除了裂縫尖端可以直接寫出特別解的顯式表示式，而其他的角落，例如 L-shaped 的角落，部份參數必須先藉由其他數值方法(牛頓法求根)求得，所以其他類型的角落特別解都比較麻煩處理，所以太多奇異特別解在計算上也是不便之處。所以針對奇異裂縫處我們著重設計出如何只使用許多項基本解及少數項的奇異特別解的計算模式而形成一個稱為 Combined Trefftz method 的計算方法。因為在有限元法[Li, 1998]、Radial basis method [Li, et. al., 2009]、及基本解法[Berger, 2005]中也使用類似合併的作法，尤其在[Berger, 2005]中僅僅使用一項奇異解加入基本解中計算，所以我們設

計加入一些奇異特別解與許多項基本解合併的 Combined Trefftz method 可以用來處理具角點與裂縫奇異問題，並得到更精確的結果，因為我們加入更多符合邊界條件的特別解於演算法中以彌補基本解在奇異裂縫處之導數不連續現像。所以如果能找到準確的結果(singularity of corner solutions)，那麼對於那些因為使用傳統數值計算方法如：FEM, FDM, and FVM 而產生收斂速度降低的情形，新的計算方法例如合併法[Li, et. al., 2009]、及我們將使用的 Combined Trefftz method，都是可以將以上傳統的數值方法修正到正常好的收斂速度。所以系統化的方式使用基本解法、特別解法或兩者合併的 Combined Trefftz method 對(mixed boundary condition)混合邊界條件、及更廣義的混合邊界條件(亦即在邊界上有 free displacement and free traction conditions)，並以 Collocation Trefftz Method 計算是這個計畫的重要成果。

研究方法：

A. Introduction to the Linear Elastostatics

For easy comprehension, we may use the vector notation as in [Li, Chu, Young, Lee, 2009] and [Lee, Young, Li, Chu, 2010]. We have the Cauchy-Navier equation of linear elastostatics:

$$\Delta\vec{w}+(\frac{1}{1-2\upsilon})\nabla(\nabla\cdot\vec{w})=0,\quad in\quad S, \tag{1}$$

Where the Poisson ratio $\upsilon=\frac{\lambda}{2(\lambda+\mu)},\quad 0<\upsilon<\frac{1}{2}.$

The strain and stress relations are also given by

$$\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\delta_{ij}\sum_{k=1}^{2}\sigma_{kk},$$

There also exist the symmetric relations:

$$\sigma_{ij}=\sigma_{ji},\ \varepsilon_{ij}=\varepsilon_{ji}.$$

Based on the above notation, the Cauchy-Navier equation is written explicitly as

$$\begin{aligned}&\mu\Delta u+(\lambda+\mu)\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right\}=0,\ in\ S,\\&\mu\Delta v+(\lambda+\mu)\left\{\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right\}=0,\ in\ S,\end{aligned} \tag{2}$$

The traction on ∂S is given by

$$\vec{\tau}(\vec{w})(x)=(\tau_x(u,v),\tau_y(u,v))^T, \tag{3}$$

where

$$\tau_x(u,v)=\sigma_x n_1+\sigma_{xy}n_2,\ \text{and}\ \ \tau_y(u,v)=\sigma_{xy}n_1+\sigma_y n_2$$

or

$$\tau_x(u,v)=\sigma_x\cos(n,x)+\sigma_{xy}\cos(n,y)$$

$$\tau_y(u,v)=\sigma_{xy}\cos(n,x)+\sigma_y\cos(n,y) \tag{4}$$

where n_1 and n_2 denote the coordinates of the outward normal to the boundary. Equation 4 can be written as:

$$\tau_x(u,v)=-\sigma_x\sin\theta+\sigma_{xy}\cos\theta\quad\text{and}\quad\tau_y(u,v)=-\sigma_{xy}\sin\theta+\sigma_y\cos\theta \tag{5}$$

Where the stress component are:

$$\sigma_x=(\lambda+2\mu)u_x+\lambda v_y,\ \text{and}\ \ \sigma_x=\lambda u_x+(\lambda+2\mu)v_y\ \ \text{and}\ \ \sigma_{xy}=\mu(u_y+v_x) \tag{6}$$

In [Mogilevskaya , 1998], [Qin, 2000], for the plane stress problem, the particular solutions are expressed as the complex functions. Denote $i=\sqrt{-1}$,

$z = x + iy = re^{i\theta}$, $\bar{z} = x - iy = re^{-i\theta}$, The particular solutions u(x, y) and v(x, y) of the plane stress equations are given by the real and imaginary parts of A_k, B_k, C_k, and D_k below, respectively,

$$\begin{aligned} A_k &= iz^k + iDkz\bar{z}^{k-1}, \\ B_k &= z^k - Dkz\bar{z}^{k-1}, \\ C_k &= i\bar{z}^k, \\ D_k &= -\bar{z}^k, \quad k = 1, 2, \ldots \end{aligned} \tag{7}$$

where $D = \dfrac{1}{3 - 4\upsilon}$. By substituting the complex $z = x + iy = re^{i\theta}$, and $\bar{z} = x - iy = re^{-i\theta}$, into Equation 7, or from [Li, Chu, Young Lee, 2009], the complete particular solutions can be given as (see [Qin, 2000]):

$$\begin{aligned} u_L &= \sum_{k=1}^{L} r^k \{a_k[-\sin k\theta + Dk\sin(k-2)\theta] + b_k[\cos k - Dk\cos(k-2)\theta] + \\ &c_k \sin k\theta - d_k \cos k\theta\} + d_0, \\ v_L &= \sum_{k=1}^{L} r^k \{a_k[\cos k\theta + Dk\cos(k-2)\theta] + b_k[\sin k + Dk\sin(k-2)\theta] + \\ &c_k \cos k\theta + d_k \sin k\theta\} + c_0, \end{aligned} \tag{8}$$

Equation 8 will be used further to satisfy the rest boundary conditions. Some preliminary procedures with free traction boundary conditions on $\theta = \pm\pi$ is outlined first as a basic research procedure for our works in deriving singular particular solution.

B. <u>Derivation of singular particular solution for the first kind boundary conditions for Linear Elastostatics</u>

(1)

According to equation 8, we will derive and find solutions to fit particular boundary conditions, such as the free traction boundary conditions, and get singular solutions under these boundary conditions. This is the most tedious step. It takes some derivation.

(1a) Use equation 8, and rewrite equation 8 by replacing k by a complex variable $v^* = \nu_k$.

(1b) Equation 8 can be written as:

$$u_L = \sum_{k=1}^{L} r^{v_k} \{a_k [-\sin v_k\theta + Dv_k \sin(v_k - 2)\theta] + b_k[\cos v_k - Dv_k \cos(v_k - 2)\theta] +$$
$$c_k \sin v_k\theta - d_k \cos v_k\theta\} + d_0,$$
$$v_L = \sum_{k=1}^{L} r^{v_k} \{a_k [\cos v_k\theta + Dv_k \cos(k-2)\theta] + b_k[\sin k + Dv_k \sin(v_k - 2)\theta] +$$
$$c_k \cos v_k\theta + d_k \sin v_k\theta\} + c_0.$$

(9)

According to [Li, Chu, Young and Lee, 2010], and [Lee, Young, Li, Chu, 2010] the particular solutions have to satisfy the given boundary condition first. For example, we have to calculate u_x, u_y, v_x, v_y of Equation 9 with respect to x and y. In [Lee, Young, Li, Chu, 2010], complex functions were used to derive traction condition. In this research we will derive particular solution directly from linear elastostatics. The derived particular solutions which satisfy the given boundary conditions will lead a system of equation about the complex variable v_k. These complex variable v_k can be solved either by numerical method, such as Newton method; or in some special corner, the singular particular solutions can be derived with explicit algebraic v_k (e.g., see Equation 11 below).

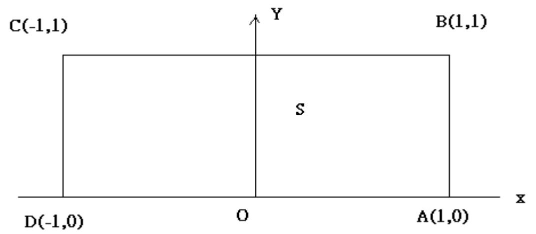

Figure 1. A Motz's Model

(1c) Design some models of crack singularity based on crack beam, models to mimic the Motz's problem) [Motz, 1947], see Figure 1 (explicitly boundary condition were not shown in the figure). Since a very accurate solution is obtained by collocation Trefftz method for this Motz's model with very high accuracy requirement in computation by MATHEMATICA [Lu, Hu, Li, 2004]. According to the free traction boundary condition at $\theta = \pm\pi$, we will have a preliminary result, such as (we give one equation for demonstration only) and assume $v = u_y = 0,\ \theta = 0$, then

$$u_L = \sum_{n=0}^{L-1} r^{n+\frac{1}{2}} \{b_{2n+1}[1 + D(n - \frac{1}{2})\cos(n + \frac{1}{2})\theta - D(n + \frac{1}{2})\cos(n - \frac{3}{2})\theta]\} + \sum_{n=1}^{L} r^n \{b_{2n}[(1 + D(n+1))\cos n\theta - Dn\cos(n-2)\theta]\} + b_0, \tag{11}$$

Also for $v_L = 0,\ at\ \theta = 0$, therefore,

$$v_L = \sum_{n=0}^{L-1} r^{n+\frac{1}{2}} \{b_{2n+1}[1 - D(n - \frac{1}{2})\sin(n + \frac{1}{2})\theta + D(n + \frac{1}{2})\sin(n - \frac{3}{2})\theta]\} + \sum_{n=1}^{L} r^n \{b_{2n}[(1 - D(n+1))\sin n\theta + Dn\sin(n-2)\theta]\},$$

The unknown coefficients have to be decided by the rest boundary traction conditions independently. Here we give only one model, and their boundary conditions are given as follows.

Model 1: Model 1 is designed as the Cauchy-Navier equation satisfying the boundary condition.

$$\begin{aligned} &\sigma_{xy} = \sigma_y = 0, \quad \text{on } \overline{OD} \\ &\mathrm{v} = \mathrm{u}_y = 0, \quad \text{on } \overline{OA} \\ &u = v_x = 0, \quad \text{on } \overline{AB} \cup \overline{CD} \\ &u = 1,\ v_x = 0, \quad \text{on } \overline{BC} \end{aligned} \tag{12}$$

(1d) On each model, different particular solutions will be obtained according to their boundary conditions and are used in the collocation Trefftz method in the following step 2 to solve the unknown parameters.

(2)

Up to this step, there are still unknown parameters need to be solved, and will be solved in the next step 3 by the collocation Trefftz method.

After completion of step 1, particular solutions of Equation 5 with different boundary conditions will be obtained, take Equation 11 as example, and define basis particular solutions, such as:

$$\begin{aligned}
&\Phi_{2n+1}(r,\theta) = r^{n+\frac{1}{2}}[(2+D(n+\frac{1}{2}))\cos(n+\frac{1}{2})\theta - D(n+\frac{1}{2})\cos(n-\frac{3}{2})\theta],\\
&\Phi_{2n}(r,\theta) = r^{n} Dn[\cos(n\theta) - \cos(n-2)\theta],\\
&\Psi_{2n+1}(r,\theta) = r^{n+\frac{1}{2}} D(n+\frac{1}{2}))[-\sin(n+\frac{1}{2})\theta + \sin(n-\frac{3}{2})\theta],\\
&\Psi_{2n}(r,\theta) = r^{n}[(2-Dn)\sin(n\theta) + Dn\sin(n-2)\theta].
\end{aligned} \tag{13}$$

Different basis other than these boundary conditions will be obtained according to different boundary condition from Equations 12, and their derivation are similar to the current procedure.

(3)

Design the collocation Trefftz Method. We choose particular solutions from step (1) and denote the solution set as V_L . Since they satisfy the governed equations 5, and some boundary conditions, the unknown parameters left in step 1, are sought by satisfying the rest boundary conditions. Here we will define the boundary energy, for example,

$$I(u,v) = \int_{\overline{BC}} [(v-1)^2 + \omega^2 u_y^2] + \int_{\overline{AB}\cup\overline{CD}} (u^2 + \omega^2 v_x^2) \tag{14}$$

where ω is some weight, usually we choose $\omega = \frac{1}{L}$, L is the number of unknown of the particular solutions.

Define the collocation Trefftz method as: To seek $(u_N, v_N) \in V_L$ such that

$I(u_L, v_L) = \min_{(u,v)\in V_L} I(u,v)$ where

$$\hat{I}(u,v) = numerical\ \ value(\int_{\overline{BC}} [(v-1)^2 + \omega^2 u_y^2] + \int_{\overline{AB}\cup\overline{CD}} (u^2 + \omega^2 v_x^2)) \tag{15}$$

We approximate the integral equation 15 by some integration rule, such as midpoint or the Gaussian quadrature. Details will not be given here.

(4)

On the other hand, we can also use equation 15 to form a collocation equation set, such as,

$$\sqrt{h}\{\sum_{n=1}^{2L} b_n \Psi_n(r_i,\theta_i)\} = \sqrt{h}, (r_i,\theta_i) \in \overline{BC} \tag{16a}$$

$$\omega\sqrt{h}\{\sum_{n=1}^{2L} b_n \frac{\partial}{\partial y}\Phi_n(r_i,\theta_i)\} = 0, (r_i,\theta_i) \in \overline{BC} \qquad (16b)$$

$$\omega\sqrt{h}\{\sum_{n=1}^{2L} b_n \frac{\partial}{\partial X}\Psi_n(r_i,\theta_i)\} = 0, (r_i,\theta_i) \in \overline{AB} \cup \overline{CD} \qquad (16c)$$

$$\sqrt{h}\{\sum_{n=1}^{2L} b_n \Phi_n(r_i,\theta_i) + b_0\} = 0, (r_i,\theta_i) \in \overline{AB} \cup \overline{CD} \qquad (16d)$$

Equation 16 forms and a square or an over-determined system. Some numerical method such as QR or least square method can be implemented to solve the unknown coefficients.

(5)

In the implementation of collocation Trefftz method, particular solutions are obtained after completion of steps 1 and 2, 3, and 4. If fundamental solutions needed to be used, then we can try to use the combined method developed by Li [Li, 2008]. We use a few terms of the singular particular solutions together with many fundamental solutions to form a combined Trefftz method [Lee, Young, Li, Chu, 2010]. The set of complete solutions can be written as the following: (take one solution as demonstration)

$$u_L = \sum_{j=1}^{N}\{a_j(-2\ln r_{PQ_j} + D\frac{1}{r_{PQ_j}^2}[(x-\xi_j)^2 - (y-\eta_j)^2] +$$

$$+ b_j 2D\frac{(x-\xi_j)\cdot(y-\eta_j)}{r_{PQ_j}^2} + c_j\frac{(x-\xi_j)}{r_{PQ_j}^2}\} + \sum_{j=1}^{L} d_{2j+1}\Phi_{2j+1}(r,\theta) + d_0,$$

$$b_{2n+1}[(2 + D(n+\frac{1}{2}))\cos(n+\frac{1}{2})\theta - D(n+\frac{1}{2})\cos(n-\frac{3}{2})\theta],$$

(17a)

$$v_L = \sum_{j=1}^{N}\{b_j(-2\ln r_{PQ_j} + D\frac{1}{r_{PQ_j}^2}[(y-\eta_j)^2 - (x-\xi_j)^2] +$$

$$+ a_j 2D\frac{(x-\xi_j)\cdot(y-\eta_j)}{r_{PQ_j}^2} + c_j\frac{(y-\eta_j)}{r_{PQ_j}^2}\} + \sum_{j=1}^{L} d_{2j+1}\Psi_{2j+1}(r,\theta) + d_0,$$

$$b_{2n+1}[(2 + D(n+\frac{1}{2}))\cos(n+\frac{1}{2})\theta - D(n+\frac{1}{2})\cos(n-\frac{3}{2})\theta],$$

(17b)

Equations 17a,b consist of two parts, one is the set of fundamental solutions, and another is a small set of singular particular solutions. But the small sub-region contains source points Q_j of fundamental solutions is estimated and experimented before a suitable region is determined. This is the so called Combined Trefftz Method.

C. Derivation of singular particular solution for the second kind boundary conditions for Linear Elastostatics (mixed type boundary conditions- the first kind)

(1)

At the initial step, we should be aware of how to use the complete set of particular solutions, i.e., equation 8 to match the boundary conditions given in the outline shown above paragraph. That means **steps 1~5 in the outline above** will be repeated for different boundary conditions. **We want to find the particular solution with mixed boundary conditions, one is free displacement and the other one is a free traction. For the free displacement boundary condition, that is,** $u = v = 0$, on $\theta = 0$ **, and the free traction boundary condition** $\tau_x = \tau_y = 0,\ on\ \theta = \Theta$ where the traction τ_x and τ_y are written in equations 5 and 6. A new particular solution from equation 9 satisfying the free displacement condition should make the equation 9 to be a simplified form as:

$$u_L = \sum_{k=1}^{L} r^{v_k} \{a_k[-2\sin v_k\theta + Dv_k(\sin(v_k-2)\theta - \sin v_k\theta)] + b_k Dv_k[\cos v_k - \cos(v_k-2)\theta]$$

$$v_L = \sum_{k=1}^{L} r^{v_k} \{a_k\, Dv_k[-\cos v_k\theta + \cos(v_k-2)\theta] + b_k[2\sin v_k\theta + Dv_k(\sin(v_k-2)\theta - \sin v_k\theta)].$$

(18)

The derivation of the new particular solution satisfying the free traction condition is the most tedious task. In [Lee, Young, Li, Chu, 2010], complex functions were used to derive traction condition. In the current project, direct derivation from linear elastostatics will be used. Hence we will need one lemma from [Li, Chu, Young and Lee, 2009].

Lemma: There exists the equalities,

$$\frac{\partial r}{\partial x}=\cos\theta,\frac{\partial r}{\partial y}=\sin\theta,\frac{\partial \theta}{\partial x}=\frac{-\sin\theta}{r},\frac{\partial \theta}{\partial y}=\frac{\cos\theta}{r}.$$

By chain rule, we should be able to derive u_x, u_y, v_x, v_y by

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and $\sigma_x, \sigma_y, \sigma_{xy}$ by equation 5 or equation 6. Details will be carried in the project.

(2)

After the completion of **step 1**, we should have an equation which describe the power $v_k=\alpha_k+i\beta_k$, for $i=1,2,\ldots,L$ derived from equation 18 under the condition that existence of nonzero solutions of the variables a_k and b_k. Based on this equation, we can then find solution of α_k & β_k either by numerical method or an explicit formula. For the numerical computation part, it is certainly workable. But, we like to find if there is explicit algebraic equation which describes these particular solutions, since it our goal to find an explicit particular solution under these mixed boundary conditions. We should be able to write the particular singular solution derived from equation 18 into the form as:

$$\begin{aligned} u_L &= \sum_{k=1}^{L} r^{v_k}\{a_k\,\Phi 1 + b_k D v_k \Phi 2\} \\ v_L &= \sum_{k=1}^{L} r^{v_k}\{a_k\,\Psi 1 + b_k \Psi 2]. \end{aligned} \qquad (19)$$

where

$$\begin{aligned} \Phi 1 &= [-2\sin v_k\theta + Dv_k(\sin(v_k-2)\theta - \sin v_k\theta)] \\ \Phi 2 &= Dv_k[\cos v_k - \cos(v_k-2)\theta] \\ \Psi 1 &= Dv_k[-\cos v_k\theta + \cos(v_k-2)\theta] \\ \Psi 2 &= [2\sin v_k\theta + Dv_k(\sin(v_k-2)\theta - \sin v_k\theta)]. \end{aligned} \qquad (20)$$

But, if we can find the relation between a_k, b_k, then we may re-write the equation 19 as:

$$u_L = \sum_{k=1}^{L} r^{\nu_k} \{a_k^1 \Phi 1 + a_k^2 \Phi 2\}$$
$$v_L = \sum_{k=1}^{L} r^{\nu_k} \{a_k^1 \Psi 1 + a_k^2 \Psi 2\}. \quad (21)$$

Equations 19 or 21 will be used in the computation for accurate numerical singular particular solutions in the models we design. Note that though the unknown in equation 19 is the same as equation 21, but if the unknown a_k^1, a_k^2 maybe complex conjugate each other, then the unknown can be further reduced, and it will be studies in the project.

(3)

After will have the new set of particular solution satisfying the free displacement condition and free traction condition as in equations 19, and 21. The unknown variables a_k^1 and a_k^2 will solved by satisfying the rest of boundary conditions in the new designed model. We will imitate the Motz's problem as in [Li, Chu, Young and Lee, 2009].

And the boundary conditions can be defined as;

$$\begin{aligned} &\tau_x = \tau_y = 0, \quad \text{on } \overline{OD} \\ &\text{v} = \text{u} = 0, \quad \text{on } \overline{OA} \\ &u = v_x = 0, \quad \text{on } \overline{AB} \cup \overline{CD} \\ &\sigma_y = -1, \ \sigma_{xy} = 0, \quad \text{on } \overline{BC} \end{aligned} \quad (22)$$

It means that on the edge $\overline{BC}$ there is a stress on the negative y direction, and no stress on the x component. Several new models should be given other than this model shown in equation 22 to test the accuracy and stability of collocation Trefftz Method.

(4)

As in the outline above, we can set up a collocation Trefftz model. We choose particular solutions from **step (3)** and denote the set as V_L. Since these particular solutions already satisfy the governed equations 1, and some boundary conditions, the unknown parameters a_k^1 and a_k^2 in equation 19, are sought by satisfying the rest boundary conditions. Here we will define the boundary energy, for example,

$$I(u,v) = \int_{\overline{AB} \cup \overline{CD}} [u^2 + \omega^2 v_x^2] + \omega^2 \int_{\overline{BC}} (u_x + 3v_y + 1)^2 + (u_y + v_x)^2] \quad (23)$$

where ω is some weight, usually we choose $\omega = \frac{1}{L}$, L is the number of unknown of the particular solutions. Define the Collocation Trefftz method as: To seek $(u_N, v_N) \in V_L$ such that

$I(u_L, v_L) = \min_{(u,v)\in V_L} I(u,v)$ where

$$\hat{I}(u,v) = numerical\ \ value\{\int_{\overline{AB}\cup\overline{CD}} [u^2 + \omega^2 v_x^2] + \omega^2 \int_{\overline{BC}} (u_x + 3v_y + 1)^2 + (u_y + v_x)^2]\}$$

(24)

On the other hand, we can also use equation 24 to form a set of collocation equation similar to equation 16a and 16b, 16c, and 16d depending on the boundary energy given in equation 24, with suitable weighting factor ω, an algebraic over-determined system is formed. Some numerical method such as QR or least square method can be implemented to solve the unknown coefficients.

(5)

After we solve the algebraic equation, we can get accurate solution, and the leading power v_1 and the first coefficient. These informations will be used in calculating Stress Intensity Factor. Such as Qin[2000]:

$$K_I = \lim_{r\to 0} \frac{\sqrt{2\pi}}{r^{v_1 - 1}} \sigma_y(r,0) \tag{25}$$

$$K_{II} = \lim_{r\to 0} \frac{\sqrt{2\pi}}{r^{v_1 - 1}} \sigma_{xy}(r,0) \tag{26}$$

D. <u>(D) Derivation of singular particular solution for the second kind boundary conditions for Linear Elastostatics (mixed type boundary conditions- the second kind)</u>

The procedure in **(c)** can also be used to calculate another set of boundary conditions. We call it mixed boundary condition of the second kind.

(1) By the experience from the previous paragraph in derivation of the singular particular solutions and forming the collocation Trefftz method, we should be now aware how to use the complete set of particular solutions to match the boundary conditions given by the model and derive the singular particular solutions. Now, we want to get the particular solution with mixed boundary conditions on the same corner edge. **We want to find the particular solution with mixed boundary conditions on the same corner edge, with a free displacement and a free traction, we call it the mixed boundary condition of the second kind.** For example,

$$u = v_y = 0; \; at \;\; \theta = 0, \tag{27}$$

and the mixed boundary condition

$$u^* = v_v^* = 0; \; at \;\; \theta = \Theta, \tag{28}$$

where $0 < \Theta \le 2\pi$, u^* and v^* are the displacements along and perpendicular to the edge $\theta = \Theta$, respectively, and υ is the external normal. A new particular solution from equation 19 satisfying equation 27 can be written as:

$$\begin{aligned} u_L &= \sum_{k=1}^{L} r^{v_k} \{a_k [-\sin v_k \theta + D v_k \sin(v_k - 2)\theta] + c_k \sin v_k \theta - d_k \cos v_k \theta\}, \\ v_L &= \sum_{k=1}^{L} r^{v_k} \{a_k [\cos v_k \theta + D v_k \cos(k-2)\theta] + c_k \cos v_k \theta + d_k \sin v_k \theta\} + c_0. \end{aligned} \tag{29}$$

We will use equation 29 to satisfy the next mixed boundary condition given at equation 28. But a transformation formula will be needed.

$$\begin{aligned} u^* &= u \cos\theta + v \sin\theta \\ v^* &= -u \sin\theta + v \cos\theta \end{aligned} \tag{30}$$

We shall have a new set of particular solutions as:

$$\begin{aligned} u^* &= \sum_{k=1}^{L} r^{v_k} \{a_k (-1 + D v_k) \sin(v_k - 1)\theta] + c_k \sin(v_k + 1)\theta\} + c_0 \sin\Theta, \\ v^* &= \sum_{k=1}^{L} r^{v_k} \{a_k (1 + D v_k) \cos(v_k - 1)\theta + c_k \cos(v_k + 1)\theta\} + c_0 \cos\Theta. \end{aligned} \tag{30-1}$$

And

$$\begin{aligned} u_v^* &= \sum_{k=1}^{L} r^{v_k - 1} \{a_k (1 - D v_k)(1 - v_k) \cos(v_k - 1)\theta] + c_k (1 + v_k) \cos(v_k + 1)\theta\}, \\ v_k^* &= \sum_{k=1}^{L} r^{v_k - 1} \{a_k (1 + D v_k)(1 + v_k) \sin(v_k - 1)\theta - c_k (1 + v_k) \sin(v_k + 1)\theta\}. \end{aligned}$$

(2) After the completion of **step 1**, we should have an equation which describe the powers $v_k = \alpha_k + i\beta_k$, for $i = 1, 2, ..., L$. Based on this equation, we can then find solution of α_k & β_k either by numerical method or an explicit formula. It is our goal to find an explicit particular solution under these mixed boundary conditions. If possible, we should be able to write the singular particular solution equation 29 into the form which should be similar to equation 11: For example, the boundary condition is given as:

$$u = v_y = 0,\ at\ \theta = 0$$

$$v = u_y = 0,\ at\ \theta = 2\pi$$

Then the singular particular solution can be written as:

$$u_L = \sum_{k=1}^{L} r^{\frac{2k-1}{4}} \{a_{2k}[-\sin(\frac{2k-1}{4})\theta + D(\frac{2k-1}{4})\sin(\frac{2k-9}{4})\theta] + c_k[\sin(\frac{2k-1}{4})\theta]\}, \tag{31}$$

$$v_L = \sum_{k=1}^{L} r^{\frac{2k-1}{4}} \{a_{2k}[\cos(\frac{2k-1}{4})\theta + D(\frac{2k-1}{4})\cos(\frac{2k-9}{4})\theta] + c_k[\cos(\frac{2k-1}{4})\theta]\}, \tag{32}$$

In addition, Equation 31 , 32 will be used in the computation for accurate singular particular solutions in the models we design. Other boundary conditions will be explored in the research by deriving explicit particular solutions.

(3)

After will have the new set of particular solution satisfying the free displacement condition and free traction condition as in equations 31 and 32. The unknown variables a_k and c_k will solved by satisfying the rest of boundary conditions in the new designed model. We will imitate the Motz's problem as in [Li, Chu, Young and Lee, 2009].

And the boundary conditions can be defined similarly as the one given in the part 1as;

$$\begin{aligned} &v = u_y = 0,\ \text{ on } \overline{OD} \\ &u = v_y = 0,\ \text{ on } \overline{OA} \\ &u = v_x = 0,\ \text{ on } \overline{AB} \cup \overline{CD} \\ &\sigma_y = -1,\ \sigma_{xy} = 0,\ \text{ on } \overline{BC} \end{aligned} \tag{32}$$

It means on the edge $\overline{BC}$ there is a stress on the negative y direction, and no stress on the x component.

(4)

Up to this stage, we can set up a collocation Trefftz model. Design the Collocation

Trefftz Method. We choose particular solutions from **step (3)** and denote the set as V_L . Since they satisfy the governed equations 1, and some boundary conditions, the unknown parameters left in **step (3)**, are sought by satisfying the rest boundary conditions. Here we will define the boundary energy, for example,

$$I(u,v) = \int_{\overline{AB}\cup\overline{CD}} [u^2 + \omega^2 v_x^2] + \omega^2 \int_{\overline{BC}} (u_x + 3v_y + 1)^2 + (u_y + v_x)^2] \tag{33}$$

where ω is some weight, usually we choose $\omega = \frac{1}{L}$, L is the number of unknown of the particular solutions.

Define the Collocation Trefftz method as: To seek $(u_N, v_N) \in V_L$ such that

$I(u_L, v_L) = \min_{(u,v)\in V_L} I(u,v)$ where

$$\hat{I}(u,v) = numerical\ value\{\int_{\overline{AB}\cup\overline{CD}} [u^2 + \omega^2 v_x^2] + \omega^2 \int_{\overline{BC}} (u_x + 3v_y + 1)^2 + (u_y + v_x)^2]\} \tag{34}$$

On the other hand, we can also use equation 34 to form a collocation equation similar to equations 16a~d, and an over-determined system is formed. Some numerical method such as QR or least square method can be implemented to solve the unknown coefficients.

(5)

After we solve the algebraic equation, we can get accurate solution, and the leading power v_1 and the first coefficient. These information will be used in calculating Stress Intensity Factor. Such as Qin[2000]:

$K_I = \lim_{r\to 0} \frac{\sqrt{2\pi}}{r^{v_1-1}} \sigma_y(r,0)$ and $K_{II} = \lim_{r\to 0} \frac{\sqrt{2\pi}}{r^{v_1-1}} \sigma_{xy}(r,0)$

Results after completion of these steps in (C) and (D):

1. It will lead to a systematically derivation not only singularity property but also singularity particular solutions of corners for two different cases, one is with free stress on one side and a clamped boundary condition on the other side; and the other is a mixed boundary conditions on the same corner edge, that is one free displacement and a free traction boundary condition on the same corner edge.

2. It will lead to a systematically analysis of fundamental solutions and singular

particular solutions for different boundary conditions.

3. Some crack models which mimic the Motz's problem will be established.

4. A combined Trefftz method is formed by many fundamental solutions and a few singular particular solutions to solve corners crack singularity of linear elastostatics.

5. Calculate Stress Intensity factor K_I and K_{II} shown in equation 26 to provide accurate values for application.

結果與討論：

這個計畫『線彈性體的奇異特別解及數值計算』的完成已經累計有四篇文章發表在 SCI 期刊上。線彈性體以 Collocation Trefftz Method 計算的文章包含[Li, Chu, Young, and Lee, 2009]、[Li, Tsai, Lee, Young 2010] 、[Lee, Young, Li, Chu, 2010]都已經在 Eng. Anal. Bound. Elem. 刊登；另外與線彈性體有關但使用不同的計算方式及其與 Collocation Trefftz Method 比較的文章『Hybrid and Collocation Trefftz Methods for traction Boundary Conditions in Linear Elastostatics (old title: Boundary Methods for Traction Conditions in Linear Elastostatics)』 [Li, Tsai, Lee, Young 2010]亦在去年十一月被 Eng. Anal. Bound. Elem. 通知接受刊登了。另外線彈性體基本解法的級數展開『New Series Expansions for Fundamental Solutions of Linear Elastostatics in 2D 』[Li, Lee, Chen, 2010]在去年十月份被 Computing 期刊通知接受，這些級數展開式對未來處理線彈性體誤差分析及推廣到重調和方程也很重要[Li, Lu, Hu, 2004]，以上兩篇文章計畫主持人皆是通訊作者。

所以我們在 Crack and corner singularity of linear elastostatics 的計算上已經累積部分心得，尤其在 Combined Trefftz Method 上我們領先其他研究者。在計畫結束時我們已經在線彈性體各種不同的邊界條件之下，有系統性的推導奇異特別解，並在不同的奇異問題中推導出代數型態的明顯特別解。對於基本解法、特別解法、及混合型法規(Combined method)，我們已經累計非常重要的計算經驗，並提供準確的斷裂強度(Stress Intensity Factor)給工程界參考。

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計畫完成與相關出版彙整圖表

<table>
<tr><td colspan="3">Proposal Title:
Singularity of Linear Elastostatics and Stokes Equations with Their Numerical Solutions</td></tr>
<tr><td colspan="3">Linear elastostatics problem with mixed boundary conditions</td></tr>
<tr><td colspan="2">項目/研究主題/研究性質</td><td>相關參考資料</td></tr>
<tr><td>Particular solution for Cauchy-Navier equations</td><td>(理論)</td><td>1. Mogilevskaya , 1998
2. Muskhelishvili, 1953</td></tr>
<tr><td>1. A set of general particular solutions.
2. Traction by complex variable formulation and expressed in summation of L terms with some unknown need to be satisfied according to certain boundary conditions.</td><td>(理論)</td><td>1. Muskhelishvili, 1953</td></tr>
<tr><td>Particular solutions near corners with mixed boundary conditions. That is a free traction boundary conditions and a free displacement on two</td><td>(理論)</td><td>1. Li, Chu, Young, and Lee 2009</td></tr>
</table>

different edges.		2. Lee, Young, Li, Chu, 2010
Set up models of crack singularity with mixed boundary conditions.	(理論)	1. Li, Chu, Young, and Lee 2009 2. Lee, Young, Li, Chu, 2010
1. Form the collocation Trefftz method, basis of singular particular solutions can be obtained. 2. A system of linear algebraic equations will be obtained, either QR or a least-square method can be used to solve the system.	(理論) (程式技術)	1. Li, Chu, Young, and Lee 2009 2. Lee, Young, Li, Chu, 2010 3. Li, 2008 4. Li, et. al., 2009
To form a combined Trefftz method by using many fundamental solutions and a few singular particular solutions.	(理論) (程式技術)	1. Lee, young, Li, and Chu, 2009 2. Lee, Young, Li, Chu, 2010 2. Li, 2008
Numerical examples	(程式技術)	1. Li, Chu, Young, and Lee 2009 2. Lee, Young, Li, Chu, 2010
Results of the works 1. **Lee, M. G**. , Young L. J., Li, Z. C. , and Chu, P.C. (2010, 3), Combined Trefftz Methods of Particular and Fundamental Solutions for Corner and Crack Singularity of Linear Elastostaatics, Engineering Analysis with Boundary Elements, Vol. 34, pp 632~654, 2010. **[SCI] (IF:1.531-1.704-5 years)** 2. Li, Z. C. , Chu, P.C., Young L. J., and **Lee, M. G**. (2010, 2), Models of corner and crack singularity of linear elastostatics and their numerical solutions. Accpted and sent for print, Engineering Analysis with Boundary Elements, Vol. 34, pp. 533~548. **[SCI]**	Journal Publication	1. 2010 2. 2010

(IF:1.531-1.704-5 years)		
第二階段 **Linear elastostatics problem with two kinds of mixed boundary conditions, one is on the same corner edge and the other is on different edges**		
項目/研究主題/研究性質		相關參考資料
Particular solution for Cauchy-Navier equations	(理論)	1. Mogilevskaya , 1998 2. Muskhelishvili, 1953
1. A set of general particular solutions. 2. Some coordinate transformation is performed. 2. Traction by complex variable formulation and expressed in summation of L terms with some unknown need to be satisfied according to certain boundary conditions.	(理論)	1. Muskhelishvili, 1953
Particular solutions near corners with mixed boundary conditions. That is a free traction boundary conditions and a free displacement on the same corner edge.	(理論)	1.Li, Chu, Young, and Lee 2009 2.Lee, Young, Li, Chu, 2010
Particular solutions near corners with mixed boundary conditions. That is a free traction boundary conditions and a free displacement on different corner edges.	(理論)	1.Li, Chu, Young, and Lee 2009 2.Lee, Young, Li, Chu, 2010
Set up models of crack singularity with two kinds of mixed boundary conditions.	(理論)	1. Li, Chu, Young, and Lee 2009 2. Lee, Young, Li, Chu, 2010
1. Form the collocation Trefftz method, basis of singular particular solutions can be obtained. 2. A system of linear algebraic equations will be obtained, either QR or a least-square method can be used to solve the system.	(理論) (程式技術)	1 Li, Chu, Young, and Lee 2009 2. Lee, Young, Li, Chu, 2010

		3.Li, 2008
To form a combined Trefftz method by using many fundamental solutions and a few singular particular solutions.	(理論) (程式技術)	1. Lee, young, Li, and Chu, 2009 2. Lee, Young, Li, Chu, 2010 3. Li, 2008
Numerical examples	(程式技術)	1.Li, Chu, Young, and Lee 2009 2. Lee, Young, Li, Chu, 2010
Results of the works 3. **Lee, Ming-Gong** and Li, Zi-Cai, *Corner and Crack Singularity of Different Boundary Conditions for Linear Elastostatics and Their Numerical Solutions*, Engineering Analysis with Boundary Elements, Vol. 36 (2012), pp. 1125-1137. (2012.03.05) [**SCI] (IF:1.531-1.704-5 years)** 4. **Ming-Gong Lee**, L.J. Young, Z.C. Li, and P.C. Chu, Mixed Types of Boundary Conditions at Corners of Linear Elastostatics and Their Numerical Solutions, Accepted by Engineering Analysis with Boundary Elements, April (2011). [**SCI] (IF:1.531-1.704-5 years)**	Journal Publications	**1. 2012** **2. 2011**

國科會補助計畫衍生研發成果推廣資料表

日期:2012/10/05

國科會補助計畫	計畫名稱: 線性彈性體與Stokes方程奇異問題與數值方法的研究
	計畫主持人: 李明恭
	計畫編號: 100-2115-M-216-001- 學門領域: 偏微分方程數值計算

無研發成果推廣資料

100 年度專題研究計畫研究成果彙整表

計畫主持人：李明恭			計畫編號：100-2115-M-216-001-				
計畫名稱：線性彈性體與 Stokes 方程奇異問題與數值方法的研究							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	1	1	100%		
		研討會論文	2	2	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	2	2	50%	人次	
		博士生	1	1	50%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	4	2	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	1	1	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以 100 字為限）

   　　□實驗失敗

   　　□因故實驗中斷

   　　□其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：■已發表 □未發表之文稿 □撰寫中 □無

   專利：□已獲得 □申請中 ■無

   技轉：□已技轉 □洽談中 ■無

   其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

   我的研究成果已經有四篇國際 SCI 期刊論文(Engineering Analysis with Boundary Elements (IF:1.531-1.704-5 years))，因為這個計劃的相關論文也有兩篇發表在重要的國際期刊(Computing,Engineering Analysis with Boundary Elements)，成果可為豐富。當然在影響力方面，在基礎研究的領域事實上無法評估實際的影響力，但是工程師在使用相關的方法或成果時就有依據的地方，所以本研究也是有應用價值的。