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Diameter, Connectivity and Edges of Graphs

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Abstract.

In the past years, we study edge number, maximum genus and decay number of graphs with given diameter and connectivity. We found that there exist some relations among edge number, connectivity and diameter. In particular, we proved that $q \geq 2p-11$ for any diameter 3 graph G with minimum degree $\delta(G)=3$, where p, q denote the edge number and vertex number of G, respectively. This result and the past research intrigue us to study the edge number of graphs with given diameter and connectivity. In this project, we prove that if G is a 3-regular $(p,q)-$graph of diameter 3, then $p \leq 16$ and $q \leq 2p-8$. We also show this bound is sharp. Finally, we conjecture that $q \geq 2p-8$ for any diameter 3 $(p,q)-$graph G with $\delta(G)=3$.

1. Preliminary

Throughout this project, a graph may have multiple edges or loops, but a multigraph contains multiple edges, but no loops, and a simple graphs contains neither multiple edges nor loops. Let the decay number of G, $\varsigma(G)$ (resp. the Betti deficiency, $\xi(G)$) be the minimum number of components (resp. odd size components) of a co-tree of a connected graph G. The invariant $\xi(G)$ was introduced to calculate the maximum genus $\gamma_M(G)$ of G by the formula $\gamma_M(G)=(\beta(G)-\xi(G))/2$, where $\beta(G)$ is the Betti number. It is clear that $\xi(G) \leq \varsigma(G)$ for any graph. In the past years, we study edge number, maximum genus and decay number with given diameter and connectivity. We find that there exist some relations among edge number, connectivity and diameter. This relations followed the study of Murty[4] in 1969. He proved the following theorem.

Theorem 1.1(Murty[4]) *If G is a 2-connected, diameter 2*

$(p,q)-$graph, then

$$q \geq 2p-5$$

After the study of Murty[4], Bollobas and Harary【3】in 1976 defined that $P(p,d,d',s)$ is the set of graphs with order p, diameter at most d and diameter at most d' if it is deleted any s vertices. Furthermore, $f(p,d,d',s)$ is defined by the minimum edge number of all graphs in

$P(p,d,d',s)$. Thus, $f(p,d,d-1,n-1)$ denotes the minimum edge number of $n-$connected, diameter d graphs. To simplify the notation, we use $f(p,d,n)$ to replace $f(p,d,p-1,n-1)$. By the definition, Theorem 1 show that $f(p,2,2)\geq 2p-5$. Next, Bollobas and Harary【3】obtained the following results.

Theorem 2 (Bollobas and Harary【3】)

$$f(p,2,n)\geq\left\{\frac{1}{2}(p-1)(n+1)-n^2+2n\right\},$$

where $\{x\}$ *denotes the least integer not less than* x.

Moreover, Bollobas[1] replaced n-connected by minimum degree n. He proved the following theorem.

Theorem 3 (Bollobas [1])

Let G *be a graph of order* d *and size* g *with minimum degree at least* $n\geq 3$ *and diameter* d.

(1) If $2\leq d\leq 2m$ for some m, then

$$q\geq\frac{p}{2}\left(n+\frac{n-1}{(n-1)^m-1}\right)-C(n,m)$$

where $C(3,m)=4^{m+1}$ and $C(n,m)=n^{m+1}$ for $n\geq 4$

(2) If $3\leq d\leq 2m+1$,then

$$q\geq\frac{p}{2}\left(n+\frac{1}{\sum_{i=1}^{m+1}(n-1)^i}\right)-(n+1)p^{\frac{m+1}{m+2}}$$

By the above theorem, $f(p,d,n)\geq 2p-4p^{\frac{2}{3}}$.

In 2004, we study the 3-connected, diameter 3 graphs, and we obtained the following result.

Theorem 4 (Tsai and Fu [10])

If G *is a diameter* 3 $(p,q)-$*graph with* $\delta(G)=3$*, then*

$$q\geq 2p-11.$$

By the above theorem, we have $f(p,3,3)\geq 2p-11$. By Theorem 1 and

Theorem 4, we find that $f(p,k,k) \geq 2p-c$, where c is a constant for $k=2,3$. For $k=4$, it is clear that $f(p,k,k) \geq 2p$. That is, $f(p,k,k) \geq 2p-c$ for some constant.

On the other hand, Murty[4] found the extremal 2-connected, diameter 2 $(p,q)-$graphs with $q \geq 2p-5$.

Theorem 5 (Murty[4])

The following statements are equivalent for a graph G.

(1) G *is an extremal 2-connected graph of diameter 2*

(2) G *is either the Petersen graph or is constructed by connecting all vertices of* K_2 *or* K_3 *to a new vertex by paths of length 2.*

In [9], we found that Theorem 1 cloud be extended. We defined that a subset A of $E(G)$ is $E-$minimal if any two different components of $G-A$ are joined by at most one edge in G.

Theorem 6 (Tsai[9])

Let G *be a 2-connected graph of diameter 2 and let* A *be an* $E-$minimal*subset of* $E(G)$. *Then*

$$m(G/A) \geq 2n(G/A)-5+i(G/A),$$

where $i(G/A)$ *is the number of components in* $G-A$ *containing at least two vertices.*

By the above theorem, Fu, Tsai and Xuong[8] found the extremal 2-connected, diameter 2 graphs with $\varsigma(G)=4$ and $\xi(G)=4$.

Theorem 7 (Fu, Tsai and Xuong[8])

Let G *be a 2-connected graph of diameter 2. Then* $\varsigma(G)=4$ *if and only if* G *is an extremal 2-connected graph of diameter 2 with loops added to vertices.*

Theorem 8 (Fu, Tsai and Xuong[8])

Let G *be a 2-connected graph of diameter 2. Then* $\xi(G)=4$ *if and only if* G *is an extremal 2-connected graph of diameter 2 at each vertex of which an odd number of loops added to vertices.*

The past research intrigue us to study the edge number of graphs with given diameter and connectivity. Therefore, we have the following objectives in this project.

(1) Find the minimum edge number of graphs with given diameter and connectivity.

(2) Find the extremal graphs with given diameter and connectivity.

2. The Main Results

In order to improve Theorem 4, we consider 3-regular graphs of diameter, we get the following theorem.

Theorem 9. *Let* G *be a 3-regular* $(p,q)-$*graph of diameter 3. Then*

$$p \le 16.$$

Proof. For any $u \in V(G)$, we denote $N_i(u) = \{v \in V(G) \mid d(u,v) = i\}$,

$i = 1,2,3$. Let w be a vertex of G and z be a vertex of $N_3(w)$ with

minimum number of $N_1(z) \cap N_2(w)$. First if $|N_1(z) \cap N_2(w)| \ge 2$, then

$|N_1(z_i) \cap N_2(w)| = 3$ for any vertex $z_i \in N_3(w)$. Thus

$\sum_{z_i \in N_3(w)} |N_1(z_i)| \le 2|N_2(w)| \le 12$ and $\sum_{z_i \in N_3(w)} |N_1(z_i)| \ge 2|N_3(w)|$. This implies

that $|N_3(w)| \le 6$ and then $p \le 16$. So it is sufficient to consider

$|N_1(z) \cap N_2(w)| = 1$. We have following three cases..

Case 1. $|N_2(z)| \le 4$. Then there are at most two vertices of $N_2(w)$ which are adjacent to two vertex of $N_3(w)$. Thus $|N_3(w)| \le 6$ and then $p \le 16$.

Case 2. $|N_2(z)| = 5$. Let y_1 be a vertex of $N_2(w)$ which is adjacent to two vertices of $N_1(w)$. We denote that $N_1(w) \cap N_1(y_1) = \{x_1, x_2\}$ and $N_1(w) \setminus N_1(y_1) = \{x_3\}$. Thus if $z_1 \in N_3(w) \cap N_1(y_1)$, then there exists a

vertex of $N_3(w) \cap N_1(z_1)$ which is adjacent to one of $N_2(z_1) \cap N_1(y_3)$.

We let this vertex be z_2. Since $d(z_2, x_i) \le 3$ for $i = 1,2$, the vertex in

$N_2(z_1) \cap N_1(y_3)$ must be adjacent to one of $N_1(x_1) \cup N_1(x_2)$. This implies that at most two vertices of $N_2(w)$ which are adjacent to two vertices of $N_3(w)$. Thus $|N_3(w)| \le 7$ and then $p \le 16$.

Case 3. $|N_2(z)| = 6$. We denote that $N(w) = \{x_1, x_2, x_3\}$ and

$N(x_1) = \{w, y_1, y_2\}$, $N(x_2) = \{w, y_3, y_4\}$, $N(x_3) = \{w, y_5, y_6\}$, as in Figure 1. Without loss of generality, let z be adjacent to y_1. Since $|N_1(z) \cap N_2(w)| = 1$, the other two neighborhoods of z are in $N_3(w)$. We let them be z_1 and z_2.

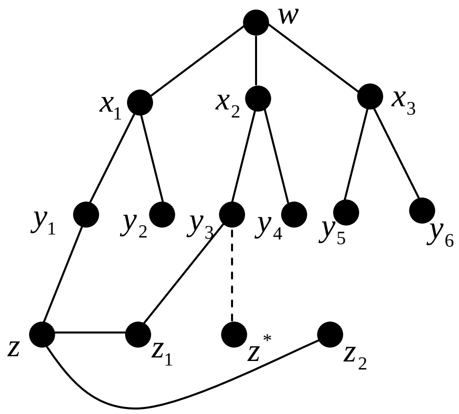

Figure 1.

For any vertex z_i, $i = 1,2$, z_i is adjacent to one vertex y_j of $N_2(w)$. If there exists one of neighborhoods of z_1 or z_2 in $N_2(w)$ which is adjacent to one vertex z^* of $N_3(w) \cap N_3(z)$, let $z_1 y_3, y_3 z^* \in E(G)$.

Assume that $z^* \notin N_i(z)$, $i = 1,2$ and z^* is not adjacent to $z_i \in N_2(z) \setminus \{y_3\}$. Since for any $z_i \in N_3(w) \cap N_1(z^*) \cap N_1(y_j)$, z_i is adjacent to at least one vertex of $\{z\} \cup N_k(z)$, $k = 1,2$, in order to keep $d(z^*, y_j) \le 3$, for all j, z^* must be adjacent to two vertices of

$\{y_1, y_2, y_5, y_6\}$. Assume that z_2 is adjacent to a vertex $z_3 \in N_3(w) \setminus \{z\}$ and z_3 is adjacent to a vertex $z_4 \in N_3(w) \setminus \{z_2\}$. Then in order to keep $d(z^*, z_i) \le 3$ for $i = 2,3,4$, z_3 is adjacent to neighborhoods of z^* or both of z_2 and z_4 are adjacent to neighborhoods of z^*. If z_3 is adjacent to neighborhoods of z^*, but z_2 and z_4 are not adjacent to y_4, then $d(x_2, z_3) > 3$. If z_3 is adjacent to neighborhoods of z^*, z_2 or z_4 is adjacent to y_4, then $d(z_i, y_j) > 3$ for some $i = 2$ or 4, $j = 1,2$ or 5,6. If z_3 is not adjacent to neighborhoods of z^*, then both of z_2 and z_4 are adjacent to neighborhoods of z^*. Thus $d(x_2, z_3) > 3$. This is a contradiction. Hence there is no $z_3 \in N_3(w) \setminus \{z\}$ and $z_4 \in N_3(w) \setminus \{z_2\}$ such that z_2 is adjacent to a vertex z_3 and z_3 is adjacent to a vertex z_4.

Now we can count $|N_3(w)|$.

$$
\begin{aligned}
|N_3(w)| &\le |\{z\}| + |N_1(z) \cap N_3(w)| + |N_2(z) \cap N_3(w)| + |N_3(z) \cap N_3(w)| \\
&\le 1 + |\{z_1, z_2\}| + |N_1(z_1) \cup N_1(z_2) \cap N_3(w)| \\
&\quad + |\{z^*\} \cup N_2(z_1) \cup N_2(z_2) \cap N_3(w)| \\
&\le 1 + 2 + 2 + 2 = 7.
\end{aligned}
$$

For otherwise, there exists no neighborhoods of z_1 or z_2 in $N_2(w)$ which is adjacent to one vertex z^* of $N_3(w) \cap N_3(z)$. Thus

$$|N_3(w)| \le |\{z\}| + |N_1(z) \cap N_3(w)| + |N_2(z) \cap N_3(w)| + |N_3(z) \cap N_3(w)|$$

$\leq 1 + |\{z_1, z_2\}| + |N_1(z_1) \cup N_1(z_2) \cap N_3(w)|$

$+ |N_2(z_1) \cup N_2(z_2) \cap N_3(w)|$

$\leq 1 + 2 + 2 + 2 = 7.$

This implies $p \leq 17$. However, G is a 3-regular graph, G has no odd order. So we have $p \leq 16$.

Since the graph in Figure 2 is a 3-regure (16, 24)-graph of diameter, the bound in Theorem 9 is sharp.

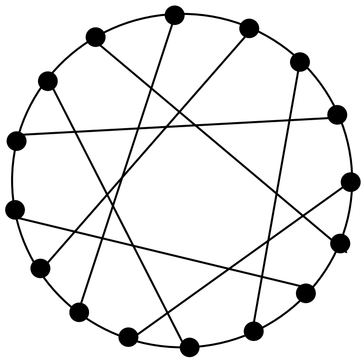

Figure 2.

By Theorem 9, we have the following corollary.

Corollary 10. *If G is a 3-regular $(p,q)-$graph of diameter 3, then*

$$q \leq 2p - 8.$$

3-regular graphs of diameter 3 have minimum number of edges among diameter 3 graphs with $\delta(G) = 3$. So we have the following conjecture.

Conjecture. *If G is a diameter 3 $(p,q)-$graph with $\delta(G) = 3$, then*

$$q \geq 2p - 8.$$

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