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Approximation of Controllable Set by Semidefinite Programming for Open-Loop Unstable Systems with Input Saturation

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***Abstract*—In order to test the efficiency of recently developed semidefinite programming (SDP), we apply SDP software package to the solution of our optimization problems. And we compare the ease of programming and the execution time for solving the problem between the classical approach (which applies a nonlinear equation solver to the Kuhn-Tucker conditions) and the SDP approach (which exploits interior-point algorithms). It is also shown that, for certain types of optimization problems, SDP is indeed very efficient. However, our examples show that SDP has limitations in solving non-convex optimization problems. It is also shown that the controllable set approximated by SDP is very efficient, however, the resulting controllable set is somewhat smaller than the set approximated under Lyapunov descent criteion.**

***Index Terms*—Controllable Set, Semidefinite Programming (SDP), Lyapunov descent criterion, Kuhn-Tucker Theorem.**

I. Introduction

Semidefinite programming (SDP) is an extension of linear programming (LP) with vector variables replaced by matrix variables and with vector elementwise non-negativity constraints replaced by matrix positive semidefiniteness constraints. Generally speaking, in semidefinite programming, one minimizes a linear function subject to the constraint that a linear combination of symmetric matrices be positive semmidefinite. A typical example of a semidefinite programming problem is

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & F(x) \succeq 0 \end{array} \qquad (1)$$

where x is a solution vector in $\Re^n$ and c is a constant vector in $\Re^n$ and F is linear with respect to *x*. We call $F(x) \succeq 0$ a *linear matrix inequality* because of linearity of F with respect to x and $F(x)$ is a square matrix. Here, $F(x) \succeq 0$ means that $F(x)$ is positive semidefinite.

Semidefinite programming has been developed both theoretically and practivally for the past few yew years, and has become a popular topic due to its efficiency in solving optimization problems with the use of interior-point methods.

Semidefinite programming also unifies several standard problems (e.g., linear and quadratic programming) and can be applied to many engineering problems (see Boyd et al. [28]), and combinatorial optimizations (see Alizadeh [26], Goemans [27]). Semidefinite programming is an important numerical tool for analysis and synthesis in control systems theory (see Vandenberghe and Boyd [29]), and many semidefinite programming problems can be solved very efficiently both in theory and practice (see [29], [26], [21], [22], and [23]).

In this paper, we applied semidefinite programming to the optimization problem of approximating the Lyapunov controllable set studied in our previous works, [32], and compare the controllable set by applying semidefinite programming software: SDPpack.

Our results show that the controllable set found by SDP is slightly smaller than the Lyapunov controllable set found by our previous work, but the commands usage are only about half of the commands written for the Lagrangian technique. Furthermore, the execution time by SDP is shorter.

II. Linear Matrix Inequality

Many problems in control and systems theory can be formulated as optimization problems in terms of linear matrix inequalities (LMIs), i.e., constraints of the form

$$F(x) \triangleq F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0, \qquad (2)$$

where $x \in \Re^m$ is the variable, and the matrices $F_i = F_i^T \in \Re^{n \times n}, i = 0, \cdots, m,$ are given symmetric constant matrices, and the inequality $F(x) \succeq 0$ represents the requirement that $F(x)$ be positive semidefinite. There are several equivalent definitions for the function of $F(x)$.

Lemma 2.1 *The following statements are equivalent for a symmetric real matrix* $F \in \Re^{n \times n}$.

1. *F is positive semidefinite.*
2. $z^T F z \geq 0, \forall z \in \Re^n$.
3. *All the eigenvalues of F are positive or zero.*
4. *There exists a real matrix $M \in \Re^{n \times n}$ such that*

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$$F = M^T M.$$

The LMI (2) is a convex constraint on x, i.e., the set $\{x \mid F(x) \succeq 0\}$ is convex. The LMI can represent a wide variety of convex constraints on x, e.g., linear inequalities, certain forms of quadratic inequalities, matrix norm inequalities, constraints arising in control theory, such as Lyapunov and convex quadratic matrix inequalities. Many conditions can be cast in the form of LMI.

III. Semidefinite Programming

We consider the optimization problem of minimizing a linear function of variable $x \in \Re^m$ subject to an LMI:

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & F(x) \succeq 0 \end{array} \qquad (3)$$

where c is a constant vector in $\Re^n$ and $F(x)$ is defined as in (1). Then we call the optimization problem (3) a *semidefinite program* (SDP). A semidefinite program is a conver optimization problem since the objective and constraints are convex: if $F(x) \succeq 0$ and $F(y) \succeq 0$, then, for all λ, $0 \le \lambda \le 1$,

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) \succeq 0.$$

There are many similarities between semidefinite programs and linear programs both in theory and practice, e.g., in duality theory, the role of complementary slackness, and availability of efficient solution techniques using interior-point methods. Foe instance, consider the following linear program (LP):

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax + b \ge 0, \end{array}$$

in which the inequality denotes a componentwise inequality. A vector v is nonnegative, $v \ge 0$, if and only if the matrix $\mathbf{diag}(v)$ is positive semidefinite $\mathbf{diag}(v) \succeq 0$. Therefore, we can express the LP as a semidefinite program with the linear matrix inequality $F(x) = \mathbf{diag}(Ax + b)$, i.e.,

$$F_0 = \mathbf{diag}(b), \quad F_i = \mathbf{diag}(a_i), \quad i = 1, \cdots, m,$$

where $A = [a_1, \cdots, a_m] \in \Re^{n \times m}$.

A convex quadratic constraint $(Ax+b)^T (Ax+b) - c^T x - d \le 0$, where $x \in \Re^m$, can be written as

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^T x + d \end{bmatrix} \succeq 0. \qquad (4)$$

The left-hand side of equation (4) depends affinely on vector x, and hence it can be expressed as a linear matrix inequality,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0,$$

where

$$F_0 = \begin{bmatrix} I & b \\ b^T & d \end{bmatrix}, \quad F_i = \begin{bmatrix} 0 & a_i \\ a_i^T & c_i \end{bmatrix}, \quad i = 1, \cdots, k.$$

Therefore, a general (convex) quadratically constrained quadratic program (QCQP) problem in $x \in \Re^m$,

$$\begin{array}{ll} \min & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, 2, \cdots, k, \end{array} \qquad (5)$$

where each f_i, $i = 0, \cdots, k$, is a convex quadratic function of the form

$$f_i(x) = (A_i x + b_i)^T (A_i x + b_i) - c_i^T x - d_i, \qquad (6)$$

or equivalently a general quadratically constrained quadratic program problem in $(x, t) \in \Re^{m+1}$,

$$\begin{array}{ll} \min & t \\ \text{subject to} & f_0(x) \le t, \\ & f_i(x) \le 0, \quad i = 1, 2, \cdots, k, \end{array}$$

can be written as

$$\begin{array}{ll} \min & t \\ \text{subject to} & \begin{bmatrix} I & A_0 x + b_0 \\ (A_0 x + b_0)^T & c_0^T x + d_0 + t \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, 2, \cdots, k, \end{array} \qquad (7)$$

We then can put the above QCQP in the SDP form:

$$\begin{array}{ll} \min & t \\ \text{subject to} & F(t, x) = F_0 + \sum_{i=1}^{m} x_i F_i + t F_{m+1} \succeq 0, \end{array} \qquad (8)$$

where the variables are $x \in \Re^m$ and $t \in \Re$.

For a non-convex optimization problem of the form,

$$\begin{array}{ll} \min & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, 2, \cdots, k, \end{array} \qquad (9)$$

where $f_i(t) = x^T A_i x + 2 b_i^T x + c_i, i = 0, 1, \cdots, k$, and the matrices A_i may be indefinite, it has been proposed by Shor and others that the lower bounds for the minimum value of $f_0(x)$ for (9) can be obtained by solving the semidefinite programming (with variables t and τ_i),

$$\begin{array}{ll} \min & t \\ \text{subject to} & \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots \\ & + \tau_k \begin{bmatrix} A_k & b_k \\ b_k^T & c_k \end{bmatrix} \succeq 0, \\ & \tau_i \ge 0, \quad i = 1, 2, \cdots, k. \end{array} \qquad (10)$$

We can easily verify that this semidefinite program yields a lower bound for the minimum value of $f_0(x)$ of (10). Suppose that x satisfies the constraints in the non-convex problem (9), i.e.,

$$f_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0,$$

for $i = 1, \cdots, k$, and that $t, \tau_1, \cdots, \tau_k$ satisfy the constraints in the semidefinite program (10). Then

$$0 \le \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left\{ \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots + \tau_k \begin{bmatrix} A_k & b_k \\ b_k^T & c_k \end{bmatrix} \right\} \begin{bmatrix} x \\ 1 \end{bmatrix}$$
$$= f_0(t) - t + \tau_1 f_1(x) + \cdots + \tau_k f_k(x)$$
$$\le f_0(x) - t.$$

Therefore, t is indeed a lower bound for the minimum value of $f_0(x)$ in (9).

IV. SOFTWARE PACKAGES FOR SEMIDEFINITE PROGRAMMING

Several software packages have been developed for the past few years for solving the semidefinite program. Here we give a brief introduction for one of the software packages applied in this paper:

1. SDP$_{\text{PACK}}$

This is a software package for Matlab 5.0 and is made by Alizadeh et al. [20].

Semidefinite-Quadratic-Linear Program (SQLP)

This package solves the primal mixed semidefinite-quadratic-linear program of the form

$$\min \quad C_S \bullet X_S + C_Q^T X_Q + C_L^T X_L$$
$$\text{subject to} \quad (A_S) \bullet X_S + (A_Q)_k^T X_Q + (A_L)_k^T X_L = b_k, k = 1, \cdots, m,$$
$$X_S \succeq 0, \quad X_Q \ge 0, \quad X_L \ge 0,$$

where X_S is a block diagonal symmetric matrix variable, with block sizes $N_1, N_2, \cdots, N_S$ respectively, each greater than or equal to two; X_Q is a block vector variable, with block sizes $n_1, n_2, \cdots, n_q$ respectively, each greater than or equal to two; and X_L is a vector of length n_0. The quantities C_Q and $(A_Q)_k, k = 1, \cdots, m$, are also vectors. The quantity $C_S \bullet X_S$ is the trace inner product ($\text{tr} C_S X_S$), i.e., $\sum_{ij} (C_S)_{ij} (X_S)_{ij}$. See [20].

V. FINDING ELLIPSOIDAL CONTROLLABLE SETS BY SEMIDEFINITE PROGRAMMING

Consider a linear time-invariant continuous-time system with input saturation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{11}$$
$$u(t) = -sat(Kx(t)), \tag{12}$$

where $A \in \Re^{n\times n}$ is a given constant matrix, $B \in \Re^{n\times m}$ is a given constant matrix, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control vector, with $u(t) = [u_1(t), \ldots, u_m(t)]$, and $sat(\cdot)$ denotes the saturation function. The one-dimensional version of the saturation function is defined by

$$sat(y) = \begin{cases} 1, & \text{if } y \ge 1 \\ y, & \text{if } y \in (-1,1) \\ -1, & \text{if } y \le -1 \end{cases}, \forall y \in \Re \tag{13}$$

and we componentwise extend its definition to the multi-dimensional version:

$$sat(y) = \begin{bmatrix} sat(y_1) \\ sat(y_2) \\ \vdots \\ sat(y_3) \end{bmatrix}, \forall y \in \Re^m. \tag{14}$$

Here we assume that A is not necessarily asymptotically stable. We also assume that the system (A, B) is linearly stabilizable. In other words, it is assumed that, without saturation, the system would be stabilizable.

Hence there exists at least one matrix K such that

$$\dot{x}(t) = Ax(t) - BKx(t) = (A - BK)x(t)$$

is asymptotically stable. Actually it is possible to select the location of the system eigenvalues (i.e., the eigenvalues of A-BK) arbitrarily. Hence we assume that matrix K has been selected so as to place the system eigenvalues in the desired location. Since $\tilde{A} = A - BK$ is Hurwitz, for every positive definite matrix $\tilde{Q}$, there exists an unique $P \in \Re^{n\times n}$ satisfying

$$\tilde{A}^T P + P\tilde{A} = -\tilde{Q},$$

and $P > 0$. Our goal is first to find an inner approximation $\Omega(P)$ of the controllable set Ω^* of our system (1) and (2) based on the quadratic Lyapunov function $V(\xi) = \xi^T P \xi$, and then to maximize the approximate controllable set $\Omega(P)$ by varying the approximation parameter P in such a way that the resulting matrix $\tilde{Q} = -(\tilde{A}^T P + P\tilde{A})$ remains positive definite.

We denote the i-th row of matrix K by $k_i, i = 1, \ldots, m$:

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}.$$

We now consider the case of a single input. Define

$$f(\xi) = A\xi - Bsat(K\xi) \tag{15}$$
$$= \begin{cases} (A - BK)\xi, & \text{if } \xi \in H_0 = \{\xi \in \Re^n \,\|\, K\xi \,|< 1\} \\ A\xi - B, & \text{if } \xi \in H_- = \{\xi \in \Re^n \mid K\xi \ge 1\} \\ A\xi + B, & \text{if } \xi \in H_+ = \{\xi \in \Re^n \mid K\xi \le -1\} \end{cases}. \tag{16}$$

Define $\tilde{V}(t) = V(x(t))$. Taking derivative of $\tilde{V}(t)$ along the trajectory $x(t)$, we obtain the following cases:

Case 1. $x(t) \in H_0$: unsaturated case, i.e., $u(t) = -Kx(t)$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= f(x(t))^T Px(t) + x(t)^T Pf(x(t)) \\ &= ((A - BK)x(t))^T Px(t) + x(t)^T P(A - BK)x(t) \\ &= x(t)^T ((A - BK)^T P + P(A - BK))x(t) \\ &= -x(t)^T \tilde{Q}x(t), \end{aligned} \tag{17}$$

Case 2. $x(t) \in H_+$: positively saturated case, i.e., $u(t) = 1$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= f(x(t))^T Px(t) + x(t)^T Pf(x(t)) \\ &= (Ax(t) + B_+)^T Px(t) + x(t)^T P(Ax(t) + B_+) \\ &= x(t)^T (A^T P + PA)x(t) + B_+^T Px(t) + x(t)^T PB_+ \\ &= -x(t)^T Qx(t) + B_+^T Px(t) + x(t)^T PB_+, \end{aligned} \tag{18}$$

where

$$Q \stackrel{\Delta}{=} -(A^T P + PA). \tag{19}$$

and

$$B_+ = B.$$

Case 3. $x(t) \in H_-$: negatively saturated case, i.e., $u(t) = -1$

$$\begin{aligned}
\frac{d}{dt}\tilde{V}(t) &= f(x(t))^T Px(t) + x(t)^T Pf(x(t)) \\
&= (Ax(t) - B_-)^T Px(t) + x(t)^T P(Ax(t) - B_-) \\
&= x(t)^T (A^T P + PA)x(t) - B_-^T Px(t) - x(t)^T PB_- \\
&= -x(t)^T Qx(t) - B_-^T Px(t) - x(t)^T PB_-,
\end{aligned} \tag{20}$$

where Q is defined as in (19) and $B_- = -B$.

Inspired by the right-hand sides (17), (18) and (20) for $\frac{d}{dt}\tilde{V}(t)$, we define

$$\begin{aligned}
g_0(\xi) &= -\xi\tilde{Q}\xi, & \xi \in H_0; \\
g_+(\xi) &= -\xi Q\xi + B^T P\xi + \xi^T PB, & \xi \in H_+; \\
g_-(\xi) &= -\xi Q\xi - B^T P\xi - \xi^T PB, & \xi \in H_-;
\end{aligned}$$

Combining these three functions into one function, we obtain

$$g(\xi) = \begin{cases} g_0(\xi) \text{ if } \xi \in H_0, \\ g_+(\xi) \text{ if } \xi \in H_+, \\ g_-(\xi) \text{ if } \xi \in H_-, \end{cases} \tag{21}$$

Observe that

$$\frac{d}{dt}\tilde{V}(t) = g(x(t)) \tag{22}$$

We want to find the maximum level set $L(r) = \{\xi \in \Re^n : V(\xi) = \xi^T P\xi \le r\}$ of the Lyapunov function V that is contained in the descent region $R_g \underline{\underline{\Delta}} \{\xi \in \Re^n : g(\xi) \le 0\}$ in which the time derivative of the Lyapunov function is negative, i.e.,

$$r^* = \max\{r \mid L(r) \subset R_g = \{\xi \in \Re^n : g(\xi) \le 0\}\}.$$

We note that, in Case 1 $\tilde{Q} > 0$ because P is selected so that $\tilde{Q} > 0$. In other words, because we use only those P that will make $\tilde{Q} = -(\tilde{A}^T P + P\tilde{A}) > 0$, the right-hand side for $\frac{d}{dt}\tilde{V}(t)$ is negative: $g_0(\xi) \le 0, \forall \xi \ne 0$. Hence $g(\xi) < 0$, $\forall \xi \in H_0 - \{0\}$. Therefore, the equilibrium point is locally asymptotically stable in H_0. However, in Case 2 and Case 3, since the open-loop system may be unstable, matrix A may not be Hurwitz. Given a positive definite matrix P that will make $\tilde{Q} > 0$, the Q defined by (9) may or may not be positive definite.

In order to satisfy the Lyapunov descent condition $g(\xi) < 0$ for a given ξ, we require that for each $\xi \ne 0$, there exists at least one control value ν satisfying $\|\nu\|_\infty \le 1$ and

$$g(\xi) = -\xi^T Q\xi \pm 2\xi^T PB\nu < 0.$$

Then the state space $\Re^n$ can be divided into the following regions:

(a) $R_0 = \left\{\xi \in \Re^n \,\middle|\, \xi^T \tilde{Q}\xi > 0\right\}$ If $\xi \in R_0$, then $g(\xi) < 0$.

(b) $R_+ = \left\{\xi \in \Re^n \,\middle|\, 2\xi^T PB < \xi^T \tilde{Q}\xi \le 0\right\}$ If $\xi \in R_+$, then set $\nu = 1$ so that $g(\xi) < 0$.

(c) $R_- = \left\{\xi \in \Re^n \,\middle|\, 2\xi^T PB > -\xi^T \tilde{Q}\xi \ge 0\right\}$ If $\xi \in R_-$, then set $\nu = -1$ so that $g(\xi) < 0$.

(d) $\Re^n - \{R_- \cup R_0 \cup R_+\}$. If $\xi \in \Re^n - \{R_- \cup R_0 \cup R_+\}$, then it is not possible to find $\nu \in [-1,1]$, such that $g(\xi) < 0$.

The approach for finding the maximal level set $L_P(c^*) = \left\{\xi \in \Re^n \,\middle|\, V(\xi) = \xi^T P\xi \le c^*\right\}$ which is contained in the union of the regions (a), (b) and (c), i.e.,
$c^* = \max\left\{c \mid L_P(c) \subset R_0 \cup R_+ \cup R_-\right\}$,
can be found by the following maximization problem

$$\begin{aligned}
c^* = \quad & \min \quad V(\xi) = \xi^T P\xi \\
& \text{subject to} \quad g_+(\xi) = -\xi^T Q\xi + B^T P\xi + \xi^T PB \ge 0, \\
& \text{and} \quad K\xi + 1 \le 0.
\end{aligned} \tag{23}$$

The above optimization problem with Lyapunov descent criterion can be solve by Kuhn-Tucker Theorem, see Wang and Mukai [32].

We now apply SDP to the above optimization problem (23).

Since $P > 0$, we can find $\sqrt{P}$ such that $P = \sqrt{P}\sqrt{P}$. Therefore, $V(\xi)$ can be put in the format of (6). But Q may not be positive definite, and thus we may not be able to decompose Q into its square roots.

Note that the optimization problem (23) is exactly in the formulation of QCQP as in (5) and (6), except the inequality constraint. We now attempt to rewrite the above optimization problem in the QCQP format (5) and (6)

$$\begin{aligned}
& \min \quad t \\
& \text{subject to} \quad x^T Px \le t, \\
& \qquad x^T Qx - 2B^T Px \le 0, \\
& \qquad K\xi + 1 \le 0.
\end{aligned}$$

We note that Q is indefinite, we need to apply the non-convex optimization technique to the optimization problem (5) and (6) to the form of (9) and (10). Recall that for a non-convex optimization problem of the form,

$$\begin{aligned}
& \min \quad f_0(x) \\
& \text{subject to} \quad f_i(x) \le 0, \quad i = 1, 2, \cdots, k,
\end{aligned}$$

where $f_i(t) = x^T A_i x + 2b_i^T x + c_i,\ i = 0, 1, \cdots, k.$ The matrices A_i can be indefinite. The lower bounds for the minimum value of $f_0(x)$ for the above optimization problem can be obtained by solving the semidefinite programming (with variables t and τ_i) in the following SDP formulation:

$$\begin{array}{ll} \max & t \\ \text{subject to} & \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \\ & + \tau_2 \begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \succeq 0, \\ & \tau_i \geq 0, \quad i = 1, 2. \end{array} \tag{24}$$

Therefore, we rewrite the optimization problem (24) into the form

$$\begin{array}{ll} \max & t \\ \text{subject to} & \begin{bmatrix} P & 0 \\ 0 & -t \end{bmatrix} + \tau_1 \begin{bmatrix} Q & -PB \\ -B^T P & 0 \end{bmatrix} \\ & + \tau_2 \begin{bmatrix} 0 & \frac{1}{2}K^T \\ \frac{1}{2}K & 1 \end{bmatrix} \succeq 0. \end{array}$$

VI. EXAMPLE

We now demonstrate the techniques for finding the controllable sets with a practical example which has been studied several times in the past [33] [34].

Example 1. Consider the double integrator, a single input plant of the form

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ -1 \leq u \leq 1. \end{cases}$$

Here, we note that the eigenvalues of the open-loop systems are found to be 0, 0. Since there are two open-loop zeros for the system, the system is open-loop unstable. Suppose that the desired eigenvalues $\lambda_1' = -\sigma + i\omega$ and $\lambda_2' = -\sigma - i\omega$ for the closed-loop system are as follows:

(i) $\sigma_1 = 1, \omega_1 = 1.$

(ii) $\sigma_2 = 2, \omega_2 = 1.$

Technique 1. Maximize the set inside the Lyapunov descent region.

$$\begin{array}{ll} \min & V(\xi) = \xi^T P \xi \\ \text{subject to} & g_+(\xi) = -\xi^T Q \xi + B^T P \xi + \xi^T PB \geq 0, \\ \text{and} & K\xi + 1 \leq 0. \end{array}$$

Technique 2. Approximate the set by semidefinite programming.

$$\begin{array}{ll} \max & t \\ \text{subject to} & \begin{bmatrix} P & 0 \\ 0 & -t \end{bmatrix} + \tau_1 \begin{bmatrix} Q & -PB \\ -B^T P & 0 \end{bmatrix} \\ & + \tau_2 \begin{bmatrix} 0 & \frac{1}{2}K^T \\ \frac{1}{2}K & 1 \end{bmatrix} \succeq 0. \end{array}$$

Figure 1 shows the sets of two approximations for the controllable set of case (i). The outer ellipse (solid line) is the Lyapunov controllable set approximated by Technique 1, in which the area is found as $A_1 = 1.5091$, while the outer ellipse (chained line) is the controllable set obtained from the non-convex optimization technique of SDP (Technique 2), in which the area is found as $A_2 = 1.1437$,

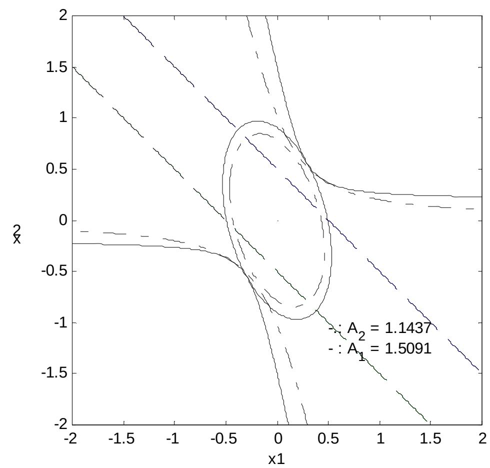

Figure 1: Comparison of two approximations for case (i)

Figure 2 shows the sets of two approximations for the controllable set of case (ii). The outer ellipse (solid line) is the Lyapunov controllable set approximated by Technique 1, in which the area is found as $A_1 = 0.3672$, while the outer ellipse (chained line) is the controllable set obtained from the non-convex optimization technique of SDP (Technique 2), in which the area is found as $A_2 = 0.2349$,

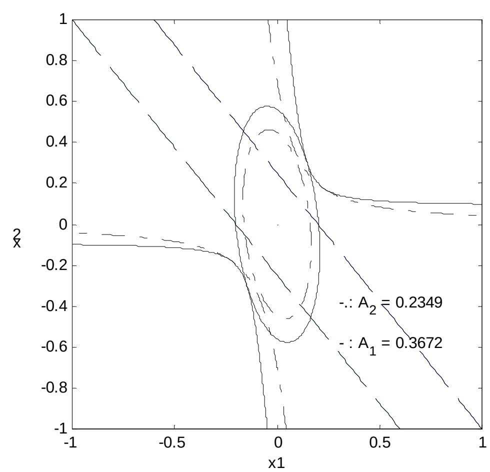

Figure 2: Comparison of two approximations for case (ii)

VII. CONCLUSION

In this paper, we applied SDP to the problems of approximating the controllable set for the open-loop unstable system with input saturation. Even with the efficiency of SDP in solving the optimization problems, our example showed that there is a limitation for applying the SDP to the problem of approximating the Lyapunov controllable set. Therefore, an inner approximation of the Lyapunov controllable set was devised. The example in this paper gave a better insight into the task of approximating the controllable set: in the two-dimensional example, as we applied SDP to solve the optimization problem, the area of inner approximation was off by about 30% of the Lyapunov controllable set; however, the command usage and executing time for the inner approximation of the Lyapunov controllable set solved by SDP were far superior to those of the conventional way of finding the Lyapunov controllable set using the Lagrangian technique.

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